Celestial Sight Reduction - Law of Cosines

By Ron Davidson, SN

Overview

A mariner typically keeps track of cruise progress by calculating and recording a deduced reckoning (DR) position from his/her last <u>known</u> (fix) position, using compass course steered, along with speed and time travelled, to calculate distance travelled using 60D = S*T and plotting the result on the active chart or plotting sheet.

Note: The DR position is an unambiguous latitude and longitude that is precisely plotted. Now for the ambiguity: *is the mariner <u>actually</u> located at the DR position?*

Following the navigation axiom "*keep a DR position and verify the DR by other means*", the mariner's task is to verify his/her actual location relative to the DR position. The "blue water" mariner, having few visible references to assist in verifying position, typically uses a sextant to measure and record the altitude (height) observed of a celestial body above the visible horizon from his/her actual location and records the measurement along with the exact time the sight was taken. Once that task is accomplished, the mariner then uses his/her DR position as a reference position along with data gleaned from the current edition of the *Nautical Almanac* to calculate what the altitude of the chosen celestial body would be if measured from the DR position. The altitude calculated using the DR position as a reference location is then compared to the altitude actually observed from the mariner's location to determine if there is a difference between the two altitudes.

If there is no difference in altitudes, the mariner can then conclude that the sextant (actual) sighting altitude was indeed taken from the DR position and the DR is verified. As you might guess, this rarely occurs. If there is a difference in the altitudes, then the mariner must rightly conclude that the actual sighting was NOT taken from the DR position but some location apart from the DR position.

The mariner's task now becomes one of determining the distance and direction from the DR position to the actual location. The process used to determine this distance offset and direction is called *Sight Reduction*. Once the distance and direction are determined, the mariner can then refer to the active chart, locate the plotted DR, and measure and plot a point on the chart at that distance and in the direction determined from the DR position and mark and label that point as an *Estimated Position*.

It is an *Estimated Position* because it was determined by only a single sextant sighting. To determine a "fix" of position, the mariner must observe, measure and record, the altitude of a second celestial body (within 20 minutes of the first sighting), then using almanac data, calculate what the altitude of the second body would be from the DR position, compare the observed and calculated altitudes, determine distance and direction from the DR position, then plot the results of both sightings on the chart to obtain a fix of position.

An Interesting Note about Nautical Almanac Data

If our Latitude and Longitude are known, we can use the almanac data for a particular day and a little arithmetic to determine any listed celestial body's altitude, azimuth, or geographic position (GP) at any second of any minute of any hour of that day from our position.

Conversely, in Celestial Navigation, we use the almanac data and a little arithmetic to determine a celestial body's altitude, azimuth, and GP to find our Latitude and Longitude at the hour, minute, and second of our sextant sighting.

The Four Altitudes

Before delving into the details of sight reduction and to alleviate confusion, an explanation of the four "altitude names" used in sight reduction is warranted.

- Sextant Altitude (Hs) – The actual reading taken from the sextant when the sighting is measured.
- Apparent Altitude (Ha) – The sextant altitude corrected for the height of our eyes above sea level (Dip) and any instrument error (Instrument Correction (Ic)) inherent in the sextant and is then used as an entering argument to extract additional correction data from the almanac.
- Observed Altitude (Ho) – The apparent altitude additionally corrected for atmospheric refraction and other data extracted from page A2 (or A3) of the almanac for the body observed.
- Calculated Altitude (Hc) – The altitude of the celestial body calculated from the DR reference position.

Sight Reduction

"Sight reduction" is the steps taken to extract almanac data relating to the celestial body chosen at the time of the sextant sighting and applying the arithmetic necessary to solve the Navigational Triangle for position. Download a USPS Sight Reduction Form to follow along.

The celestial bodies used for navigation are: Sun, Moon, Venus, Mars, Jupiter, Saturn, and the 57 "selected" stars compiled in the almanac.

The general procedure for reducing a sight is the same for each body however, the steps followed for reducing a sight taken of the Sun, Moon, a planet, or a star are each a bit different.

Sight Corrections Applied

The altitude measured with the sextant at the time of sighting is called the sextant altitude (Hs) and must have corrections applied to convert it to our apparent altitude (Ha) then used as the entering argument into the Nautical Almanac tables on page A2, et al to extract the appropriate data.

- The error corrections to be applied to Hs to arrive at Ha are :
  - Instrument Error – The inherent error of the sextant – Found on the manufacturer's calibration certificate that accompanies the sextant.
  - Index Error – If the index mirror and horizon mirror of the sextant are not precisely parallel there is Index Error. We measure this error each time we use the sextant by setting the sextant to 0° 0.0' then sighting the natural horizon and adjusting the micrometer until the two halves of the horizon coincide and read and record the reading as Index Error. This error may be positive or negative to be added to or subtracted from Hs.
  - Dip – This correction accounts for our eyes being above the horizon at the time of the sighting and also accounts for terrestrial refraction. It is *always* subtracted from Hs.

Using the apparent altitude (Ha (App Alt in the almanac page A2)) we can extract correction data from the almanac. The almanac data will account for other corrections such as atmospheric refraction and semi-diameter, etc. and be applied to Ha to finally arrive at our observed altitude (Ho).

Sea Time

The positions of celestial bodies are tabulated in the *Nautical Almanac* according to Universal Time, formerly called Greenwich Mean Time (GMT), and abbreviated as UT.

The mariner must convert their local ship's time to UT before extracting additional data from the Almanac.

If you're unfamiliar with this process, a more detailed explanation of time can be found in my Celestial Navigation Primer.

Reducing a Sun Sight – Using the Altitude – Intercept Method

Let's start with an example: You may use the Online Almanac set for January 5th 2017 to follow along although the data may be slightly different than that in the printed almanac.

On January 5, 2017, from a DR position of 47° 24.0' N 122° 20.1' W at watch time 12-14-59 a sight is taken on the lower limb of the Sun and a sextant altitude (Hs) of 19° 55.1' is measured. The height of eye above sea level is 15 feet; sextant Index Error (IE) is 01.5' off the arc (negative). There is no watch error.

Our Observed Sight

We record the date, time, watch error, the DR, the name (and limb if Sun or Moon) of the body observed, the height of eye above sea level, the sextant measurement (Hs), and Index Error.

Next, we apply corrections:

Sextant Altitude - Hs	19° 55.1'	
Index Correction – Ic	+01.5	(Index Correction is the opposite sign of Index Error)
	19° 56.6'	
Dip Correction	-3.8'	(from the Dip table on page A2 of the almanac)
Apparent Altitude – Ha	**19° 52.8'**	

Using Ha as the entering argument into App Alt column of page A2 of the almanac, find the App Alt entries that bracket our Ha and we extract the Main correction for a Lower Limb Sun sighting from the Sun table, Oct-Mar column as +13.6'

Apparent Altitude – Ha	19° 52.8'
Main correction	+13.6
Observed Altitude – Ho	**20° 06.4'**

Ho is the altitude measured from our actual location. We'll now compare Ho to what the altitude of the Sun would be from our DR position by calculating that altitude (Hc) using the latitude and longitude of our DR position as a reference along with data about the Sun, at the time of our sighting, extracted from the almanac.

Our Calculated Sight

First, our watch time of 12-14-59 must be converted to UT. Our DR longitude is 122° 20.1' (122.335°) W. The Sun moves across the sky East to West at 15° per hour. 122.335 ÷ 15 = 8.15566 rounded to 8 hours west of Greenwich. Time at the Greenwich Meridian is therefore 8 hours later than our local time. UT = 12-14-59 + 8 = 20-14-59.

We open the almanac to 2017 JANUARY 4, 5, 6 (WED., THURS., FRI). From the right-hand daily page day 5 Sun column at whole hour 20 we extract and record the GHA of the Sun as 118° 35.0' and the Dec of the Sun as 22° 30.8' S and the *d* correction at the bottom of the Sun column as 0.3.

Next we turn to the INCREMENT AND CORRECTIONS page for minute 14 and follow down the column to second 59 and extract the change in GHA during 14m 59s as 3° 44.8'. Also on the 14m page from the *v or d Corr* column we find the *d* of 0.3 and extract the correction as 0.1'.

We now combine the whole hour GHA with the change in GHA for 14m 59s to get total GHA:

118° 35.0'
+03° 44.8'
122° 19.8' Total GHA.

We now combine our DR longitude (- if west, + if east) with our Total GHA to determine our Local Hour Angle (LHA).

Total GHA	122° 19.8'
Longitude	122° 20.1'W (-)
LHA	359° 59.7'

LHA is the longitudinal angular difference between our DR (local) longitude and the longitude of the GP of the Sun. We see in this example that this difference is near zero, meaning the Sun's GP is nearly on our DR meridian of longitude, making this Sun sight a "*Noon Sight for Latitude*".

Now we turn to Declination. Earlier we extracted a *dec* of 22° 30.8' S. We also extracted a *d corr* of 0.1'. Is the *d corr* to be added or subtracted? Return to the right-hand daily page for 2017 JANUARY 4, 5, 6 (WED., THURS., FRI) and find the *dec* for the next whole hour (21) of UT. Is the dec increasing further South or less South? It is 22° 30.5' S or less South, so the *d corr* is subtracted.

Dec	22° 30.8' S
d corr	-0.1
Total Dec	22° 30.7' S

We now have enough information to solve for our calculated altitude Hc using the Law of Cosines formulas.

$$\sin H_C = (\cos LHA * \cos Lat * \cos Dec) + (\sin Lat * \sin Dec)$$

$$\cos Z = (\sin Dec - (\sin Lat * \sin H_C)) + (\cos Lat * \cos H_C)$$

Convert the degrees and minutes of LHA, DR Latitude, and Declination to degrees to a precision of 5 decimal places. *Note: If DR Latitude and Declination have the same name (both N or both S) we'll enter dec as positive, otherwise negative.*

LHA	359° 59.7’ = 359.99500°
DR Latitude	47° 24.1’N = 47.40000°
Declination	22° 30.7’S = -22.51167°

Solving the first formula results in a sin Hc of 0.343468413. Using the sin-1 function of our calculator results in an Hc of 20.08833° or 20° 05.3’. We now compare the Ho to Hc:

Ho	20° 06.4’ altitude actually measured
Hc	20° 05.3’ altitude calculated using DR as reference position
Intercept	01.1’

The intercept tells us that at the time of our sight we were 1.1 nautical miles from our DR position. Because Ho is greater than Hc, we must have been 1.1 nautical miles closer to the Sun’s GP than our DR position. What is the azimuth (bearing) to the Sun’s GP? We must solve formula #2.

Solving the second formula results in a cosine Z of -1. Using the cos-1 function on our calculator results in Z = 180°. The Sun’s GP true azimuth (Zn) is Zn = Z (in this example). Zn = 180°. So in this example the Sun’s azimuth (Zn) is 180° T from our DR position. The Sun’s GP is due south. See the table below for determining Zn (True Azimuth) from Z. Note also that, in this example, either DR Lat N formula works because the Sun’s GP is on our DR meridian of longitude.

		If LHA	
		<180°	>180°
DR Lat	N	Zn = 360° - Z	Zn = Z
	S	Zn = 180° + Z	Zn = 180° - Z

Plot our Estimated Position

Find our DR on our plotting sheet and plot a location 1.1nm from our DR on azimuth 180°T TOWARD (closer to (Ho > Hc)) the GP (due south in this example) and that plot becomes our estimated position!

Reducing a Star Sight

A sextant sighting is taken of the star Deneb on February 12, 2017, at 18-00-30 from a DR position of 47° 24.0’N 122° 20.1’W. Sextant altitude is 25° 57.5’; height of eye is 15 feet; IE is 01.5’ off the arc; there is no watch error.

Just as we did with the Sun sight, we record the date, time, watch error, the DR, the name (and limb if Sun or Moon) of the body observed, the height of eye above sea level, the sextant measurement (Hs), and Index Error.

Next, we apply corrections:

Sextant Altitude - Hs	25° 57.5’
Index Correction – Ic	+01.5 (Index Correction is the opposite sign of Index Error)
	25° 59.0’
Dip Correction	-3.8’ (from the Dip table on page A2 of the almanac)
Apparent Altitude – Ha	**25° 55.2’**

Using Ha as the entering argument into App Alt column of page A2 of the almanac, find the App Alt entries that bracket our Ha and we extract the Main correction from the STARS AND PLANETS table as -2.0’

Apparent Altitude – Ha	25° 55.2’
Main correction	-2.0’
Observed Altitude – Ho	**25° 53.2’**

Again, our watch time of 18-00-30 must be converted to UT. UT = 18-00-30 + 8 = 26-00-30 = 02-00-30 the next day (13 Feb).

Determining the GHA of a star is a bit different. The GHA of a Star = GHA ARIES + SHA Star (GHA ✰ = GHA ♈ + SHA ✰). On the left-hand daily page for 13 February find the ARIES column and extract GHA at whole hour 02 on the 13th of 173° 18.1’. Also extract and record the SHA of Deneb from the STARS table of 49° 30.2’ and Deneb’s declination of 45° 20.5’ N.

Turn to the INCREMENT AND CORRECTIONS page for minute 0 and find second 30. Extract the change in GHA from the ARIES column of 0° 07.5’.

Now combine the GHA ARIES (♈) at 0200 with the changes of GHA for 30 seconds and the SHA of Deneb. Then combine with SHA Deneb (±360 if necessary)

173° 18.1’ hour 0200
+00° 07.5’ change in 30 sec
173° 25.6’ GHA ♈.
049° 30.2’ SHA ✰
222° 55.8’ Total GHA ✰

We now combine our DR longitude (- if west, + if east) with our Total GHA to determine our Local Hour Angle (LHA).

Total GHA	222° 55.8’
Longitude	122° 20.1’W (-)
LHA	100° 35.7’ W

We record the declination of Deneb as a total dec of 45° 20.5’ N (there is no *v or d corr* for stars).

Convert the degrees and minutes of LHA, DR Latitude, and Declination to degrees to a precision of 5 decimal places. *Note: If DR Latitude and Declination have the same name (both N or both S) we’ll enter dec as positive, otherwise negative.*

LHA	100° 35.7’ = 100.59500°
DR Latitude	47° 24.1’N = 47.40000°
Declination	45° 20.5’S = 45.34167°

Solving the Law of Cosines formulas results in a sin Hc of 0.43611776 thus an Hc of 25° 51.4’ and a cos Z of 0.640109699 thus a Z of 50.2° and a Zn of 310° T. Deneb’s GP is west of our longitude.

Comparing Ho to Hc:

Ho	25° 53.2’ altitude actually measured
Hc	25° 51.4’ altitude calculated using DR as reference position
Intercept	01.8’

Plot our Estimated Position

Find our DR on our plotting sheet and plot a location 1.8 nm from our DR on azimuth 310°T TOWARD (closer to) the GP (Ho > Hc) and that plot becomes our estimated position!

Reducing a Planet Sight

On February 15, 2017, from a DR position of 47° 24.0'N 122° 20.1'W at 18-05-00, a sight of Mars is taken. The height of the eye is 15 feet; IE is 01.5' off the arc; Hs is 34° 41.5'; there is no watch error.

Determine Ha and Ho.

Apply IE & Dip to Hs to determine Ha = 34° 39.2'.

Extract the main correction from page A2 of the almanac STARS AND PLANETS table (-1.4) and *Additional Corr* for Mars (+0.1) for a total correction of -1.3 and a Ho of 34° 37.9.

Convert our time of 18-05-00 to UT. UT = 18-05-00 + 8 = 26-05-00 or 02-05-00 next day (16 Feb).

From the left-hand daily page for 16 Feb MARS column extract GHA Mars at 0200 of 148° 18.7' and Dec Mars of 05° 22.6' N with v of 0.8 and d of 0.7.

From INCREMENTS AND CORRECTIONS page for minute 5 second 00 record the change in GHA of 01° 15' and v or d corr of 0.1' for v and 0.1 for d.

Determine Total GHA of Mars by adding GHA 0200, change in GHA, and v corr to get 164° 34.6'

Combine with DR longitude to determine LHA of 42° 14.4' W.

Combine declination with d corr for Total Dec of 05° 22.7' N

Solving the Law of Cosines formulas results in a sin Hc of 0.567891728 thus an Hc of 34° 36.2' and a cos Z of -0.58212297 thus a Z of 125.6° and a Zn of 234° T. Mars' GP is west of our longitude.

Comparing Ho to Hc:

Ho	34° 37.9'	altitude actually measured
Hc	34° 36.2'	altitude calculated using DR as reference position
Intercept	01.7'	

Plot our Estimated Position

Find our DR on our plotting sheet and plot a location 1.7 nm from our DR on azimuth 234°T TOWARD (closer to) the GP (Ho > Hc) and that plot becomes our estimated position!

Reducing a Moon Sight

Altitude corrections for the moon are more complex than those for the other bodies. The correction table for the moon (located in the back of the Nautical Almanac, pages xxxiv and xxxv) has two parts. The upper part is entered with ha, as for the sun, star, and planet tables. Another factor, called horizontal parallax (HP), is needed to enter the bottom part of the table. Because the moon's position relative to the earth changes rapidly, horizontal parallax values are tabulated for hourly intervals in the daily pages of the Almanac.

On March 04, 2017, from a DR position of 47° 24.0'N 122° 20.1'W at 18-20-30, a sight of the Moon is taken. The height of the eye is 15 feet; IE is 01.5' off the arc; Hs is 58° 02.5'; there is no watch error.

Determine Ha and Ho.

Apply IE & Dip to Hs to determine Ha = 58° 00.2'.

Convert our time of 18-20-30 to UT. UT = 18-20-30 + 8 = 26-20-30 or 02-20-30 next day (5 Mar).

Record GHA Moon from the right-hand daily page for 5 Mar at 0200 along with a v of 7.0, the declination of the Moon as 16° 40.4' N with a d of 5.5 and HP of 59.3.

Using Ha as the entering argument, extract the main correction from page xxxv of the almanac ALTITUDE CORRECTION TABLES 35° - 90° - MOON table (+40.6) and *Additional Corr* (bottom ½ of table) for Lower Limb (L) HP 59.3 (+6.3) for a total correction of 46.9 and a Ho of 58° 47.1'.

- Apparent altitude (58°00.2´) is used to enter the table, so find the column that includes your value of ha this is the column headed 55°-59°. These numbers refer to degrees of apparent altitude. Follow that column down until you come to 58°.
- Minutes of apparent altitude are listed at the sides of the table. You are looking for 00.2´. Follow across the table from 00´ and you find the value of +40.6´. The next entry, for 10´, has a value of +40.5´. Since 00.2´ is closer to 00´ than 10´, the main altitude correction value for 00.2´ is +40.6'. (You must interpolate to find the main altitude correction). The main altitude correction for the moon is always positive.
- In addition to this main correction, moon sights also require an additional correction. For the additional moon correction, use the bottom part of the table. The columns on each side are headed HP, and the inner columns are headed L and U. The L is for lower-limb sights, the U for upper-limb sights. In the same column as before (55°-59°), drop to the lower part of the table.
- Go down the L side of the column until you bracket the HP value of 59.3´ you got from the daily page. You will find an HP of 59.1´, which corresponds to a value of 6.1´. Below that is an HP of 59.4', which corresponds to a value of 6.4´. Interpolate between these bracketed values to obtain the additional moon correction for HP = 59.3´, which is +6.3´. These additional corrections are always positive.

From INCREMENTS AND CORRECTIONS page for minute 20 second 30 record the change in GHA of 04° 53.5' and v or d corr of +2.4' for v and +1.8' for d.

Determine Total GHA of the Moon by adding GHA 0200, change in GHA, and v corr to get 129° 21.4'

CombineTotal GHA with DR longitude to determine LHA of 07° 01.3' W.

Combine declination with d corr for Total Dec of 16° 42.3 N'

Solving the Law of Cosines formulas results in a sin Hc of 0.855034533 thus an Hc of 58° 45.8' and a cos Z of -0.974370065 thus a Z of 167° and a Zn of 193° T. The Moon's GP is west of our longitude.

Comparing Ho to Hc:

Ho	58° 47.1'	altitude actually measured
Hc	58° 45.8'	altitude calculated using DR as reference position
Intercept	01.3'	

Plot our Estimated Position

Find our DR on our plotting sheet and plot a location 1.3 nm from our DR on azimuth 193°T TOWARD (closer to) the GP (Ho > Hc) and that plot becomes our estimated position!

Difficulties

Perhaps the most difficult aspect of learning Sight Reduction is remembering how to extract the myriad corrections and tabular data from the almanac for sightings of the Sun, Moon, Planets, and stars. It becomes easier only with practice.

Advantages of Law of Cosines Method

Using the Law of Cosines Method of Sight Reduction requires that you have aboard only a current Nautical Almanac and a scientific calculator along with your standard plotting materials and tools.

Two other common methods of Sight Reduction require the use of a current Nautical Almanac along with either Pub 229 SIGHT REDUCTION TABLES FOR MARINE NAVIGATION (six volumes at ~$25.00 each) or Pub 249 SIGHT REDUCTION TABLES FOR AIR NAVIGATION (three volumes at ~$25.00 each). These publications each contain pre-calculated whole degree solutions of the Navigational Triangle and require slight manipulations of your DR latitude and longitude to achieve whole degrees of DEC, LAT, or LHA.